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On ANR(stratifiable)-spaces

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In metric spaces, the closed embedding theorem of Kuratowski-Wojdyslawski plays an important role in the development of retract theory. In generalized metric spaces, for a paracomplex (i.e. Hyman's M-space) X, Hyman [11] proved that X can be embedded in an AR(paracomplex)-space as a closed subset. For a stratifiable space X, Cauty [5] constructed a space Z(X), and proved that Z(X) is stratifiable and X is AR(stratifiable) (resp. ANR (stratifiable)) if and only if X is a retract (resp. neighborhood retract) of Z(X). However, it is not known whether Z(X) is ANR (stratifiable). In previous paper [16], we constructed a space E(X) for a space X, and showed that a stratifiable space X can be embedded in the AR(stratifiable)-space E(X) as a closed subset.

In this paper, in section 1 we briefly give some contents in [16], and prove some applications of the closed embedding theorem. In section 2, we consider some problems which arise in and around ANR(stratifiable)-spaces.

Throughout this paper, we assume that all spaces are regular and all maps are continuous. For the definitions of AR, ANR, AE and ANE, see Hu [10].

1. The closed embedding theorem and some applications

For a full simplicial complex, we introduced the <u>locally convex topology</u> in [15]; i.e. the strongest topology, which is locally convex, contained in the Whitehead topology. The introduction of this topology is valid by the fact that a full simplicial complex with the Whitehead topology need not be locally convex (cf. [9; pp. 416, 4.3]). Note that a full simplicial complex (with the Whitehead topology) with countable vertices is locally convex (cf. [6; Lemma 4.4]). By using the locally convex topology, we can construct $E(X)$ for a space X as follows:

CONSTRUCTION 1.1 ([16; 3.1]). Let X be a space. $A(X)$ denotes the full simplicial complex with the locally convex topology which has all points of X as the set of vertices. Let i be a canonical bijection from the 0-skeleton of $A(X)$ onto X. Then $E(X)$ is the set $A(X)$ equipped with the topology generated by sets U such that

(C1) U is open in $A(X)$ and $i(U \cap X)$ is open in X, and

(C2) U is convex in $A(X)$.

By (C1), it is clear that X is closed in $E(X)$. Note the difference of $E(X)$ and Cauty's $Z(X)$ [5]. We obtain the following:

THEOREM 1.2([16;Theorem 3.4]). If X is stratifiable, $E(X)$ is AR(stratifiable).

Next, the notion of L-space was introduced in [14]. This is defined as follows: A Hausdorff space is called an <u>L-space</u> if it can be mapped onto some stratifiable space by a perfect map. As easily seen by this definition, L-spaces have some analogous properties as paracompact M-spaces. For example:

THEOREM 1.3([14;1.3]). L-spaces are precisely the homeomorphic images of closed subsets of product $S \times I^{\tau}$, where S is stratifiable, $I = [0,1]$ and τ is an arbitrary cardinal.

Now, we give some applications of the closed embedding theorem. The following theorem is an analogous result of [19; Theorem 4].

THEOREM 1.4. Let Q be a closed hereditary class consisting of normal spaces. If every ANR(stratifiable)-space is ANE(Q), then every ANR(L)-space is ANE(Q).

Proof. Let X be any ANR(L)-space. Then by Theorem 1.3, X is a closed subset of product $S \times I^{\tau}$ where S is a stratifiable space and $\tau = w(X)$. By Theorem 1.2, S is considered as a closed subset of an AR(stratifiable)-space E(S). Therefore, we can assume that X is a closed subset of product $E(S) \times I^{\tau}$. Since $E(S) \times I^{\tau}$ is an L-space, X is a neighborhood retract of $E(S) \times I^{\tau}$. Since E(S) is ANE(Q) by assumption and I^{τ} is clearly AE(Q), $E(S) \times I^{\tau}$ is ANE(Q). Therefore X is ANE(Q). This completes the proof.

For Q satisfying the condition in this theorem, see Theorem 2.3 and Problem 2.5 in later section.

For pairs of ANR's, see [13], and for closed embeddings of pairs, see [18].

THEOREM 1.5. Every pair (X,X_0) of stratifiable spaces, where X_0 is a closed subset of X, has a closed embedding in a pair of AR(stratifiable)'s.

Proof. As easily seen from Construction 1.1, $(E(X), E(X_0))$ is a required pair of AR(stratifiable)'s.

Note that, in [18;Theorem 1.1], metric case was considered.

2. Problems

In general, the following problem naturally arises.

PROBLEM 2.1. For any two classes $\mathcal{Q}$ and $\mathcal{Q}'$ with $\mathcal{Q} \subset \mathcal{Q}'$, is an ANR($\mathcal{Q}$)-space ANR($\mathcal{Q}'$)?

For example, the case $\mathcal{Q}$ = metric has been studied as follows:

THEOREM 2.2. (1)(Dowker [8]) Every ANR(metric)-space is ANR(collectionwise normal and perfectly normal).

(2)(Lisica [12]) Every ANR(metric)-space is ANR(paracompact M).

(3)(Mardešić and Šostak [14]) Every ANR(metric)-space is ANR($\mathcal{L}$).

For the case $\mathcal{Q}$ = stratifiable, the following result was shown so far as I know.

THEOREM 2.3(Borges [1],[2]) Every ANR(stratifiable)-space is ANR(linearly stratifiable).

This theorem is easily seen by [1;Theorem 4.1 and 4.4], [2;Theorem 2.1] and [5;Theorem 1.8].

In connection with Theorem 2.2(2) and (3), the following problem was posed in [14].

PROBLEM 2.4(Mardešić and Šostak). Is an ANR(stratifiable)-space ANR($\mathcal{L}$)?

For this problem, a partial answer was proved in [14;Theorem 2.4 and 2.5]. In connection with Theorem 2.3 and Problem 2.4, we pose

PROBLEM 2.5. Find a class $\mathcal{Q}$ containing stratifiable spaces such that every ANR(stratifiable)-space is ANR($\mathcal{Q}$).

The class of all paracompact σ-spaces is not such a class (cf. [7;Example 2] and [19;Example 2]). In connection with this, Tsuda [19] posed

PROBLEM 2.6(Tsuda). If an ANR(stratifiable)-space X is ANR(paracompact σ), then is X metrizable?

Finally, let X and Y be two spaces, A a closed subset of X and $f:A \longrightarrow Y$ a map. It is well known [10;pp. 178] that if X, A and Y are ANR(metric)'s, then the adjunction space $X \cup_f Y$ is ANR (metric) provided that it is metrizable. This result was essentially proved in successive stages by Borsuk [3], Whitehead [20] and Hanner [9]. For attempt to generalize this theorem, Hyman [11] proved the case of paracomplex spaces. Cauty [4] announced the case of stratifiable spaces, but his proof was false. (This was pointed out by San-nou [17].) So, the case of stratifiable spaces is still open:

PROBLEM 2.7. Let X and Y be two stratifiable spaces, A a closed subset of X and $f:A \rightarrow Y$ a map. If X, A and Y are ANR (stratifiable)'s, is the adjunction space $X \cup_f Y$ ANR(stratifiable)?

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